

New Inequalities for Tomograms in the Probability Representation of Quantum States

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Abstract

New inequalities for symplectic tomograms of quantum states and their connection with entropic uncertainty relations are discussed within the framework of the probability representation of quantum mechanics.

keywords: tomographic map, symplectic tomography, entropic uncertainty relations, inequalities for tomograms.

1 Introduction

There exists several equivalent formulations of quantum mechanics (see, for example, review [1]). Recently a new formulation of quantum mechanics, which is called the probability representation of quantum mechanics, was introduced [2, 3]. Within the framework of this formulation, the quantum states are described by standard probability distributions instead of wave functions or density matrices. The probability representation of quantum

mechanics is equivalent to all other representations but it is more convenient for considering some class of quantum problems where one can use the well-elaborated mathematical tools of the probability theory.

The Shannon entropy [4] is the functional characteristics of any probability distribution and it was used to introduce the tomographic entropy for continuous variable [5] and for analytic signals [6], as well as for spin tomographic probabilities [7]. The probability representation was shown [8, 9, 10] to realize a new version of the quantization procedure based on the star-product formalism. The star-product approach and different properties of tomographic entropies within the quantum-information framework were studied in [11] where Rényi [12] entropy was considered using the spin-tomographic probabilities (spin tomograms).

There exist inequalities for Shannon entropies associated with probability distributions of the position and momentum (see, for example, review [13]). Recently [14] new entropic uncertainty relations based on properties of Rényi [12] entropy were found. The entropic uncertainty relations were used to obtain new inequalities for tomographic probability distributions for continuous variables (called symplectic tomograms) [15].

The aim of this work is to give a review of new entropic uncertainty relations and to obtain new inequalities for tomograms of quantum states for the case of several modes.

The paper is organized as follows.

In Section 2 we discuss the known entropic uncertainty relations for the one-mode and multimode cases. In Section 3 we review the properties of tomographic entropies [5, 6, 7]. In Section 4 we consider the integral inequalities for symplectic tomograms in the one-mode case. In Section 5 we study the the integral inequalities for both optical and symplectic tomograms of multimode quantum states. The conclusions are presented in Section 6.

2 Entropic Uncertainty Relations

If the quantum state of a particle is described by a wave function $\psi(x)$ in the position representation (or a wave function $\tilde{\psi}(p)$ in the momentum representation), the Shannon entropies S_x and S_p connected with two probability distributions $|\psi(x)|^2$ and $|\tilde{\psi}(p)|^2$ are given by the integrals:

$$S_x = - \int |\psi(x)|^2 \ln |\psi(x)|^2,$$

$$(1) \quad S_p = - \int |\tilde{\psi}(p)|^2 \ln |\tilde{\psi}(p)|^2.$$

The entropies satisfy the entropic uncertainty relation [16, 17, 13]:

$$S_x + S_p \geq \ln \pi e. \quad (2)$$

In the case of density matrices $\rho(x, y)$ and $\rho(p_x, p_y)$ of the quantum state (given in the position and momentum representations, respectively), the Shannon entropies are defined as follows:

$$(3) \quad \begin{aligned} S_x &= - \int \rho(x, x) \ln \rho(x, x) dx, \\ S_p &= - \int \rho(p, p) \ln \rho(p, p) dp. \end{aligned}$$

These entropies satisfy the same inequality (2).

In the case of multimode states, the entropies

$$(4) \quad \begin{aligned} S_{\vec{x}} &= - \int \rho(\vec{x}, \vec{x}) \ln \rho(\vec{x}, \vec{x}) d\vec{x}, \\ S_{\vec{p}} &= - \int \rho(\vec{p}, \vec{p}) \ln \rho(\vec{p}, \vec{p}) d\vec{p} \end{aligned}$$

satisfy the entropic uncertainty relation with extra factor

$$(5) \quad S_{\vec{x}} + S_{\vec{p}} \geq N \ln \pi e,$$

where N is the number of modes.

In fact, the entropic uncertainty relations (2) and (5) can be interpreted as constraints for density matrices. These constraints are connected with the positivity conditions of the density operator of any quantum state.

3 Tomograms and Tomographic Entropies

In [2] the new formulation of quantum mechanics was suggested. Within the framework of this formulation, the quantum state described by the tomographic-probability distribution $w(X, \mu, \nu)$ (called symplectic tomogram) relates to a density operator $\hat{\rho}$ by the formula [18]

$$(6) \quad w(X, \mu, \nu) = \text{Tr } \hat{\rho} \delta(X - \mu \hat{q} - \nu \hat{p}).$$

The inverse transform reads [19]

$$\hat{\rho} = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp [i(X - \mu\hat{q} - \nu\hat{p})] dX d\mu d\nu, \quad (7)$$

where \hat{q} and \hat{p} are the position and momentum operators, respectively, and X , μ , and ν are real variables. The variable X is a random position measured in a reference frame in the phase space labeled by two real parameters $\mu = s \cos \theta$ and $\nu = s^{-1} \sin \theta$. The angle θ is the rotation angle of the axis in the phase space and the scaling parameter s determines a new scale in the reference frame. Thus, one has the nonnegativity condition

$$w(X, \mu, \nu) \geq 0 \quad (8)$$

and the normalization condition of the tomographic-probability density

$$\int w(X, \mu, \nu) dX = 1. \quad (9)$$

If $s = 1$, the tomogram is called optical tomogram and it is used for measuring the quantum states of photons [20]

$$w(X, \theta) = \text{Tr } \hat{\rho} \delta(X - \hat{q} \cos \theta - \hat{p} \sin \theta). \quad (10)$$

One has

$$w(X, \mu = \cos \theta, \nu = \sin \theta) = w(X, \theta). \quad (11)$$

The symplectic tomogram satisfies the homogeneity condition

$$w(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|} w(X, \mu, \nu). \quad (12)$$

Since the optical tomogram $w(X, \theta)$ and symplectic tomogram $w(X, \mu, \nu)$ are standard probability densities, the Shannon definition was used in [5, 6] to introduce the tomographic entropies

$$S(\mu, \nu) = - \int w(X, \mu, \nu) \ln w(X, \mu, \nu) dX \quad (13)$$

and

$$S(\theta) = - \int w(X, \theta) \ln w(X, \theta) dX. \quad (14)$$

The von Neuman entropy of quantum state

$$S_N = -\text{Tr } \hat{\rho} \ln \hat{\rho} \quad (15)$$

is equal to zero for all pure quantum states.

The tomographic entropies $S(\theta)$ and $S(\mu, \nu)$ distinguish different pure states.

The homogeneity property of tomogram (12) yields the following property of tomographic entropy [21]

$$S\left(\sqrt{\mu^2 + \nu^2} \cos \theta, \sqrt{\mu^2 + \nu^2} \sin \theta\right) - \frac{1}{2} \ln(\mu^2 + \nu^2) = f(\theta). \quad (16)$$

This means that effectively the tomographic entropy depends on angle variable only.

4 Tomographic Entropic Uncertainty Relation for One Mode

Recently [15, 21] new inequalities were obtained for tomographic entropies and tomograms of quantum states for continuous variables. We present here these inequalities for the one-mode case. As it was shown in [6] the tomogram of quantum state (6) can be considered as the probability distribution of position for the state of “artificial quantum harmonic oscillator” evolving from some initial state $\hat{\rho}(0)$ to the state $\hat{\rho}(t)$. In view of this observation, the periodic-in-time motion of the oscillator provides the change of the position probability density into the momentum probability density after evolving one quarter of the vibration period. Thus, the entropies and their inequalities (2) can be calculated for tomograms of any quantum state providing the following inequality relation:

$$S(\theta) + S(\theta + \pi/2) \geq \ln \pi e. \quad (17)$$

This inequality means the integral condition for optical tomogram of quantum state

$$\int [w(X, \theta) \ln w(X, \theta) + w(X, \theta + \pi/2) \ln w(X, \theta + \pi/2)] dX + \ln \pi e \leq 0. \quad (18)$$

The optical tomogram was measured in the experiments with photons [20] and now inequality (17) can be used for extra check of the experimental data obtained.

5 Inequalities with Extra Parameters for Tomograms

In [14] the new uncertainty relation was obtained for Rényi entropy related to the probability distributions for position and momentum of quantum state with density operator $\hat{\rho}$. The uncertainty relation reads

$$\begin{aligned} & \frac{1}{1-\alpha} \ln \left(\int_{-\infty}^{\infty} dp [\rho(p, p)]^{\alpha} \right) + \frac{1}{1-\beta} \ln \left(\int_{-\infty}^{\infty} dx [\rho(x, x)]^{\beta} \right) \\ & \geq -\frac{1}{2(1-\alpha)} \ln \frac{\alpha}{\pi} - \frac{1}{2(1-\beta)} \ln \frac{\beta}{\pi}, \end{aligned} \quad (19)$$

where positive parameters α and β satisfy the constrain

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2. \quad (20)$$

Using the same argument that we employed to obtain inequality (18), one arrived at the condition for optical tomogram [21]

$$\begin{aligned} & \frac{q-1}{q} \ln \left\{ \int_{-\infty}^{\infty} dX \left[w \left(X, \theta + \frac{\pi}{2} \right) \right]^{1/(1-q)} \right\} \\ & + \frac{q+1}{q} \ln \left\{ \int_{-\infty}^{\infty} dX [w(X, \theta)]^{1/(1+q)} \right\} \\ & \geq \frac{1}{2} \left\{ \frac{q-1}{q} \ln [\pi(1-q)] + \frac{q+1}{q} \ln [\pi(1+q)] \right\}, \end{aligned} \quad (21)$$

where the parameter q is defined by $\alpha = (1-q)^{-1}$.

Below we present a new inequality for symplectic tomogram in the one-mode case. It reads

$$\begin{aligned} & \frac{q-1}{q} \ln \left\{ \int_{-\infty}^{\infty} dX \left[w \left(X, -\sqrt{\mu^2 + \nu^2} \sin \theta, \sqrt{\mu^2 + \nu^2} \cos \theta \right) \right]^{1/(1-q)} \right\} \\ & + \frac{q+1}{q} \ln \left\{ \int_{-\infty}^{\infty} dX \left[w \left(X, \sqrt{\mu^2 + \nu^2} \cos \theta, \sqrt{\mu^2 + \nu^2} \sin \theta \right) \right]^{1/(1+q)} \right\} \\ & \geq \frac{1}{2} \left\{ \frac{q-1}{q} \ln [\pi(1-q)] + \frac{q+1}{q} \ln [\pi(1+q)] \right\}. \end{aligned} \quad (22)$$

This inequality can be interpreted as a generalization of the inequality (18) extended from optical tomogram to symplectic tomogram.

In view of the inequality for Rényi entropy adopted from [14], the above condition for tomogram of quantum state can be generalized for the multi-mode case as well. Then for symplectic tomogram of quantum state with density operator $\hat{\rho}$ defined as

$$\begin{aligned} & w(X_1, X_2, \dots, X_N, \mu_1, \mu_2, \dots, \mu_N, \nu_1, \nu_2, \dots, \nu_N) \\ &= \text{Tr} [\hat{\rho} \delta(X_1 - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1) \delta(X_2 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2) \cdots \delta(X_N - \mu_N \hat{q}_N - \nu_N \hat{p}_N)] \end{aligned} \quad (23)$$

where \hat{q}_k and \hat{p}_k ($k = 1, 2, \dots, N$) are position and momentum operators, respectively, we obtain

$$\begin{aligned} & \frac{q-1}{q} \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[w \left(X_1, X_2, \dots, X_N, -\sqrt{\mu_1^2 + \nu_1^2} \sin \theta_1, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. -\sqrt{\mu_N^2 + \nu_N^2} \sin \theta_N, \sqrt{\mu_1^2 + \nu_1^2} \cos \theta_1, \dots, \sqrt{\mu_N^2 + \nu_N^2} \cos \theta_N \right) \right]^{1/(1-q)} \right\} \\ & + \frac{q+1}{q} \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[w \left(X_1, X_2, \dots, X_N, \sqrt{\mu_1^2 + \nu_1^2} \cos \theta_1, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. \sqrt{\mu_N^2 + \nu_N^2} \cos \theta_N, \sqrt{\mu_1^2 + \nu_1^2} \sin \theta_1, \dots, \sqrt{\mu_N^2 + \nu_N^2} \sin \theta_N \right) \right]^{1/(1+q)} \right\} \\ & \geq \frac{N}{2} \left\{ \frac{q-1}{q} \ln [\pi(1-q)] + \frac{q+1}{q} \ln [\pi(1+q)] \right\}. \end{aligned} \quad (24)$$

For optical tomogram $w(X_1, \dots, X_N, \theta_1, \dots, \theta_N)$, the inequality reads

$$\begin{aligned} & \frac{q-1}{q} \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} [w(X_1, X_2, \dots, X_N, \theta_1, \dots, \theta_N)]^{1/(1-q)} \right\} \\ & + \frac{q+1}{q} \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} [w(X_1, X_2, \dots, X_N, \theta_1 + \pi/2, \dots, \theta_N + \pi/2)]^{1/(1+q)} \right\} \\ & \geq \frac{N}{2} \left\{ \frac{q-1}{q} \ln [\pi(1-q)] + \frac{q+1}{q} \ln [\pi(1+q)] \right\}. \end{aligned} \quad (25)$$

Inequalities (24) and (25) are saturated for Gaussian tomograms. In the limit $q \rightarrow 0$, they become entropic uncertainty relations found in [15, 21].

6 Conclusions

To conclude, we summarize the main results of our study.

The new uncertainty relations were reviewed within the framework of the probability representation of quantum mechanics. The entropic uncertainty relations have the form of integral condition for tomograms of quantum states which contain the complete information on the states. The new inequality containing extra parameter were obtained for some integral expressions containing the quantum state tomograms on the base of recently found [14] uncertainty relations for Rényi entropy of quantum states. The conditions for the one-mode and multimode optical tomograms are of particular interest since these tomograms are directly measured in quantum-optics experiments [20]. We hope to get analogous new inequalities for tomograms depending on discrete variables.

Acknowledgments

M.A.M. thanks the Organizers of the International Workshop “Nonlinear Physics. Theory and Experiment. IV” (Gallipoli, Lecce, Italy, 2006) for kind hospitality and the Russian Foundation for Basic Research for Travel Grant No. 06-02-26768.

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